Semisimilar Solutions of the Unsteady Compressible Laminar Boundary-Layer Equations

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The theory of semisimilar solutions, previously developed for unsteady two-dimensional incompressible laminar boundary-layer flows, is extended to the case of compressible flows. A study is made of several flows for which semisimilar solutions are possible with a view of investigating the nature of compressible unsteady separation. The solutions obtained indicate that the Moore-Rott-Sears model for unsteady boundary-layer separation is valid for compressible as well as incompressible flow. In addition, the effects of Mach number and wall temperature on the moving separation point are determined.

Introduction

In recent years, there has been significant interest in the behavior of unsteady boundary layers, particularly for flows approaching separation. This interest arises in part as a result of the importance of unsteady boundary layers on the performance of helicopter blades, compressors, and turbines. Peak performance of such devices occurs when operated at conditions where the flow is near separation, or, in the case of helicopter blades, when separation occurs during a portion of the operating cycle.

It has been known for some time that the boundary-layer characteristics corresponding to unsteady separation are not the same as the characteristics of steady separation. The early studies of Rott,[1] Moore,[2] and Sears[3] led to the development of a model for unsteady boundary-layer separation which has become known as the Moore-Rott-Sears model. In this model, unsteady separation is characterized by the vanishing of both shear and velocity at an interior point of the boundary layer as seen in a coordinate system which is moving with separation. Moore[2] and Sears and Telionis[4] also argue that in the solutions of the boundary-layer equations a singularity occurs at the unsteady separation point.

In recent years, the results of a number of analytical investigations have verified the Moore-Rott-Sears model for laminar incompressible flow, both with respect to the physical characteristics associated with separation and the existence of a separation singularity. This verification has been possible only for the case of ustream-moving separation. These investigations include the work of Telionis et al.,[5] Williams and Johnson,[6,7] Telionis and Tsahalis,[8] and Williams.[9]

One of the more serious difficulties encountered in investigations of unsteady boundary layers, separating or not, is the added mathematical complexity arising from the addition of a new independent variable, time, to the boundary-layer problem. To overcome this difficulty, Telionis et al.[5,8] employed a full three-dimensional finite difference technique with a special differencing that involves marching in time, while at each time step the boundary-layer equations are solved iteratively over the entire length of the attached boundary layer. Williams and Johnson[6,7] and Williams,[9] on the other hand, employ the technique of semisimilar solutions in which the three independent variables are scaled into two new variables so that the problem can be solved employing fast, accurate methods developed for and tested on two-dimensional problems. There are, of course, limitations on the external velocity distributions which can be treated this way because there is not a semisimilar counterpart for every real external flow. Nevertheless, this technique has proven very useful in investigating the physical nature of unsteady boundary-layer separation.

The purpose of the present work is twofold. First, the theory of semisimilar solutions is developed for two-dimensional, unsteady, compressible boundary layers. Second, the newly developed theory is applied to several problems with a view to determining the effects of compressiblity on laminar unsteady separation, at least for the case in which separation moves upstream.

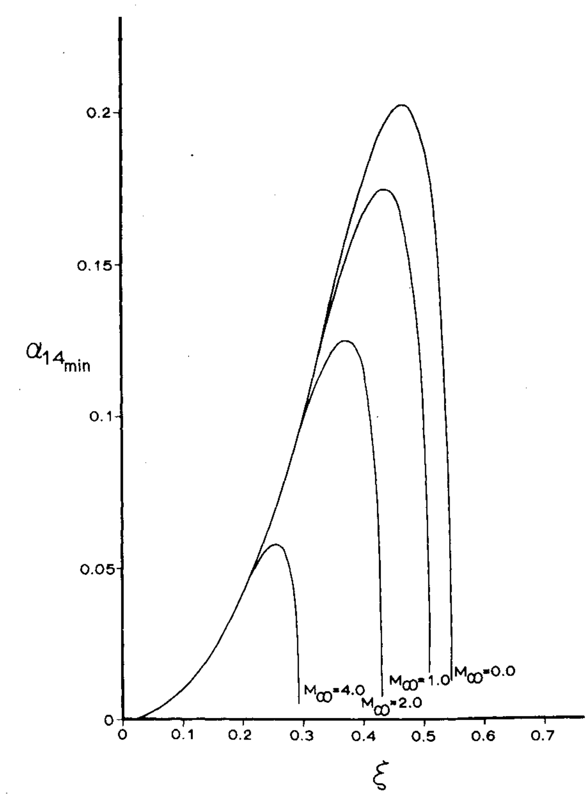

Fig. 1 Variation of the minimum value of α_{14} with ξ; unsteady Tani flow.

Received Dec. 27, 1983; revision received March 21, 1984.

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Analysis

The equations of continuity, momentum, and energy for an unsteady, thin, compressible, laminar boundary layer are

$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho u}{\partial x}+\frac{\partial\rho v}{\partial y}=0 \tag{1}$$

$$\rho\left\{\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right\}=-\frac{\partial p}{\partial x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{2}$$

$$\rho\left\{\frac{\partial H}{\partial t}+u\frac{\partial H}{\partial x}+v\frac{\partial H}{\partial y}\right\}=\frac{\partial p}{\partial t}+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\mu}{2}\frac{\partial u^2}{\partial y}\right) \tag{3}$$

Here x and y are the coordinates along and normal to the surface, u and v the respective velocity components, t the time, T the temperature, ρ the density, p the pressure, and k and μ the thermal conductivity and viscosity. Further, H is the total enthalpy given by

$$H=c_pT+u^2/2$$

and it is assumed that the gas is both calorically and thermally perfect so that c_p is a constant and $p=\rho RT$, where R is the gas constant. Finally, the subscript δ refers to conditions at the outer edge of the boundary layer. The boundary conditions necessary for the solution of these equations are

$$u(x,0,t)=v(x,0,t)=0$$

$$H(x,0,t)=H_0(x,t)\quad\text{or}\quad\frac{\partial H}{\partial y}(x,0,t)=0$$

$$\lim_{y\to\infty}u(x,y,t)=u_\delta(x,t)$$

$$\lim_{y\to\infty}H(x,y,t)=H_\delta=\text{const}$$

where the two possible boundary conditions on H at the wall correspond to either the wall temperature being prescribed or the adiabatic wall case. The pressure, velocity, and total enthalpy at the upper edge of the boundary layer are related through the Euler and energy equations at the outer edge of the boundary layer:

$$\rho_\delta\left\{\frac{\partial u_\delta}{\partial t}+u_\delta\frac{\partial u_\delta}{\partial x}\right\}=-\frac{\partial p}{\partial x}$$

$$\rho_\delta\left\{\frac{\partial H_\delta}{\partial t}+u_\delta\frac{\partial H_\delta}{\partial x}\right\}=-\frac{\partial p}{\partial t}$$

As noted in the Introduction, the solution of Eqs. (1-3) is complicated by the fact that there are three independent variables (x,y,t). While there are a variety of techniques available for solving boundary-layer problems in two variables, the techniques available for solving problems in three variables are quite limited. In addition, we expect, from incompressible results, that as separation is approached, regions of reverse flow may be encountered near the wall. If this is the case, special schemes (e.g., upwind differencing) must be employed to overcome the singular parabolic nature of the problem. These difficulties are eliminated in the technique of semisimilar solutions in which the three independent variables are scaled into two. With this in mind, we introduce the new scaled coordinates ξ and η defined by

$$\xi=\xi(x,t),\qquad \eta=\frac{1}{\sqrt{\nu_0}g(x,t)}\int_0^y\frac{\rho}{\rho_0}\mathrm{d}y \tag{4}$$

a nondimensional stream function $f(\xi,\eta)$ defined by

$$\psi=\sqrt{\nu_0}u_\delta(x,t)g(x,t)f(\xi,\eta) \tag{5}$$

and a nondimensional total enthalpy defined by

$$\theta=H(x,y,t)/H_\delta \tag{6}$$

The velocity components are related to the stream function by

$$\rho u=\rho_0\frac{\partial\psi}{\partial y}=\rho u_\delta\frac{\partial f}{\partial\eta},\qquad \rho v=-\rho_0\left\{\frac{\partial\psi}{\partial x}+\frac{\partial}{\partial t}\int_0^y\frac{\rho}{\rho_0}\mathrm{d}y\right\}$$

In terms of the new variables, the momentum equation becomes

$$\frac{\partial^3f}{\partial\eta^3}+\frac{1}{\lambda}\frac{\partial\lambda}{\partial n}\frac{\partial^2f}{\partial\eta^2}+\frac{(e+d)}{\lambda}f\frac{\partial^2f}{\partial\eta^2}+\frac{d}{\lambda}\left\{\frac{\rho_\delta}{\rho}-\left(\frac{\partial f}{\partial\eta}\right)^2\right\}$$
$$+\frac{a}{\lambda}\left\{\frac{\rho_\delta}{\rho}-\frac{\partial f}{\partial\eta}\right\}+\frac{b}{2\lambda}\eta\frac{\partial^2f}{\partial\eta^2}-\frac{c}{\lambda}\frac{\partial^2f}{\partial\xi\partial\eta}$$
$$+\frac{h}{\lambda}\left\{\frac{\partial^2f}{\partial\eta^2}\frac{\partial f}{\partial\xi}-\frac{\partial f}{\partial\eta}\frac{\partial^2f}{\partial\eta\partial\xi}\right\}=0 \tag{7}$$

and the energy equation becomes

$$\frac{\partial^2\theta}{\partial\eta^2}+\frac{1}{\lambda}\frac{\mathrm{d}\lambda}{\mathrm{d}\eta}\frac{\partial\theta}{\partial\eta}+(Pr-1)m\left[\left(\frac{\partial^2f}{\partial\eta^2}\right)^2+\frac{\partial f}{\partial\eta}\frac{\partial^3f}{\partial\eta^3}\right.$$
$$\left.+\frac{\partial f}{\partial\eta}\frac{\partial^2f}{\partial\eta^2}\frac{1}{\lambda}\frac{\partial\lambda}{\partial\eta}\right]+\frac{Pr}{\lambda}(d+e)f\frac{\partial\theta}{\partial\eta}+\frac{b}{\lambda}\frac{\eta}{2}Pr\frac{\partial\theta}{\partial\eta}-\frac{cPr}{\lambda}\frac{\partial\theta}{\partial\xi}$$
$$+\frac{Pr}{\lambda}h\left(\frac{\partial\theta}{\partial\eta}\frac{\partial f}{\partial\xi}-\frac{\partial f}{\partial\eta}\frac{\partial\theta}{\partial\xi}\right)+\frac{Pr}{\lambda}\left(\frac{\rho_\delta}{\rho}+\theta\right)n$$
$$+\frac{Pr}{\lambda}\theta\left(1-\frac{\partial f}{\partial\eta}\right)\ell=0 \tag{8}$$

where

$$a=\frac{g^{*2}}{u_\delta^*}\frac{\partial u_\delta^*}{\partial t^*} \tag{9a}$$

$$b=\frac{\partial g^{*2}}{\partial t^*} \tag{9b}$$

$$c=g^{*2}\frac{\partial\xi}{\partial t^*} \tag{9c}$$

$$d=g^{*2}\frac{\partial u_\delta^*}{\partial x^*} \tag{9d}$$

$$e=\tfrac{1}{2}u_\delta^*\frac{\partial g^{*2}}{\partial x^*} \tag{9e}$$

$$h=u_\delta^*g^{*2}\frac{\partial\xi}{\partial x^*} \tag{9f}$$

$$m=U_\infty^2u_\delta^{*2}/2H_\infty H_\delta^* \tag{9g}$$

$$\lambda=\frac{\rho\mu}{\rho_0\mu_0} \tag{9h}$$

$$\frac{\rho_\delta}{\rho}=\frac{\theta-m\left(\dfrac{\partial f}{\partial\eta}\right)^2}{1-m} \tag{9i}$$

$$n = \frac{g^{*2}}{H_\delta^* \rho_\delta^*} \frac{\partial p^*}{\partial t^*} \tag{9j}$$

$$\ell = \frac{g^{*2} u_\delta^*}{H_\delta^*} \frac{\partial H_\delta^*}{\partial x^*} \tag{9k}$$

Here the external velocity u_δ, the scaling function g, the scaled x coordinate and time have been normalized so that

$$u_\delta^* = \frac{u_\delta}{U_\infty}, \quad H_\delta^* = \frac{H_\delta}{H_\infty}, \quad g^{*2} = g^2 \frac{U_\infty}{\ell}, \quad x^* = \frac{x}{\ell}, \quad t^* = \frac{tU_\infty}{\ell}$$

where U_∞, ℓ, and H_∞ are the conditions at a suitable reference state.

The boundary conditions applicable to the solutions of Eqs. (7) and (8) are

$$f(\xi,0) = \frac{\partial f}{\partial \eta}(\xi,0) = 0, \qquad \lim_{\eta \to \infty} \frac{\partial f}{\partial \eta}(\xi,\eta) = 1$$

$$\theta(\xi,0) = \theta_w \text{ or } \frac{\partial \theta}{\partial \eta}(\xi,0) = 0, \qquad \lim_{\eta \to \infty} \theta(\xi,\eta) = 1 \tag{10}$$

where again the two possible boundary conditions at the wall correspond to either a prescribed total enthalpy at the wall (prescribed wall temperature) or an adiabatic wall case.

For semisimilar solutions to exist, coefficients a, b, c, d, e, h, ℓ, m, and n must be functions of ξ alone. We note that coefficients a, b, c, d, e, and h appear in the compressible case just as in the incompressible case. The extension of the theory to compressible flow only introduces new coefficients $\ell(\xi)$, $m(\xi)$, and $n(\xi)$. Further, coefficients a, b, c, d, e, and h are related by three ordinary differential equations, just as in the incompressible case; namely,

$$(a+b)d = 2ae + cd' - a'h \tag{11a}$$

$$2ae = 2ce' - hb' \tag{11b}$$

$$h(a+b+c') = c(h'+2e) \tag{11c}$$

where the primes denote differentiation with respect to ξ.

Solution of the set of Eqs. (7), (8), (9a-f), and (11) is, in general, quite difficult. The nine functions a, b, c, d, e, h, ℓ, m, and n are related by Eqs. (11) so that only six of these are arbitrary. Ideally, one would like to solve the problem directly for a prescribed external velocity $u_\delta(x,t)$ by solving for the appropriate scaling functions $g(x,t)$ and $\xi(x,t)$ subject to the constraints of Eqs. (11). Once the functions a, b, c, d, e, h, ℓ, m, and n are known, the solution of Eqs. (7) and (8), subject to boundary conditions (10), is straightforward using well-developed and accurate numerical techniques. Unfortunately, however, the direct method of solution is not possible and, as in the incompressible case, one is forced to solve the problem indirectly. In the indirect method of solution, one assumes the form of external velocity distribution and the scaling functions $g^*(x^*,t^*)$ and $\xi(x^*,t^*)$ and obtains the coefficients a, b, c, d, e, h, ℓ, m, and n or, alternately, one assumes the form of six of the coefficients a, b, c, d, e, h, ℓ, m, and n and obtains the other coefficients by solution of Eqs. (11). We choose here to follow the former path. Using the solution of Ref. 6 as a guide, we assume $u_\delta^* = u_\delta^*(\xi)$, $g^{*2} = x^*/u_\delta^*$ and $\xi = x^*/n(t^*)$. In the compressible case, however, it is also necessary to prescribe the thermodynamic process in the external stream. In the present case, we assume the external flow is isentropic so that

$$\frac{p_\delta}{p_0} = \left(\frac{\rho_\delta}{\rho_0}\right)^\gamma$$

where subscript 0 refers again to the suitable reference state. From Eq. (9c), in order for c to be a function of ξ alone, $n(t^*)$ must be a linear function of t, therefore, we choose $n(t^*) = 1 - \beta t^*$. The scaled variables η and ξ are then, in final form:

$$\eta = y\sqrt{U_\infty u_\delta^*}/\sqrt{\ell \nu x^*} \tag{12a}$$

$$\xi = x^*/(1-\beta t^*) \tag{12b}$$

With u_δ^* a function of ξ alone, the definition of $m(\xi)$ implies that H_δ^* is a function of ξ alone and, hence, T^* and P^* become functions of ξ alone. Under these conditions the coefficients become

$$a(\xi) = \frac{\beta \xi^2}{u_\delta^{*2}} \frac{du_\delta^*}{d\xi} \tag{13a}$$

$$b(\xi) = -a(\xi) \tag{13b}$$

$$c(\xi) = \frac{\beta \xi^2}{u_\delta^*} \tag{13c}$$

$$d(\xi) = \frac{\xi}{u_\delta^*} \frac{du_\delta^*}{d\xi} \tag{13d}$$

$$e(\xi) = (1 - d(\xi))/2 \tag{13e}$$

$$h(\xi) = \xi \tag{13f}$$

$$m(\xi) = (\gamma-1)M_\infty^2 u_\delta^{*2}/H_\delta^*[1+(\gamma-1)M_\infty^2] \tag{13g}$$

$$p^{*(\gamma-1)/\gamma} = T_\delta^* = C - (\gamma-1)M_\infty^2 \left\{\frac{u_\delta^{*2}}{2} + \beta \int \xi du_\delta^*\right\} \tag{13h}$$

$$H_\delta^* = \frac{C - (\gamma-1)M_\infty^2 \beta \int \xi du_\delta^*}{1 + [(\gamma-1)/2]M_\infty^2} \tag{13i}$$

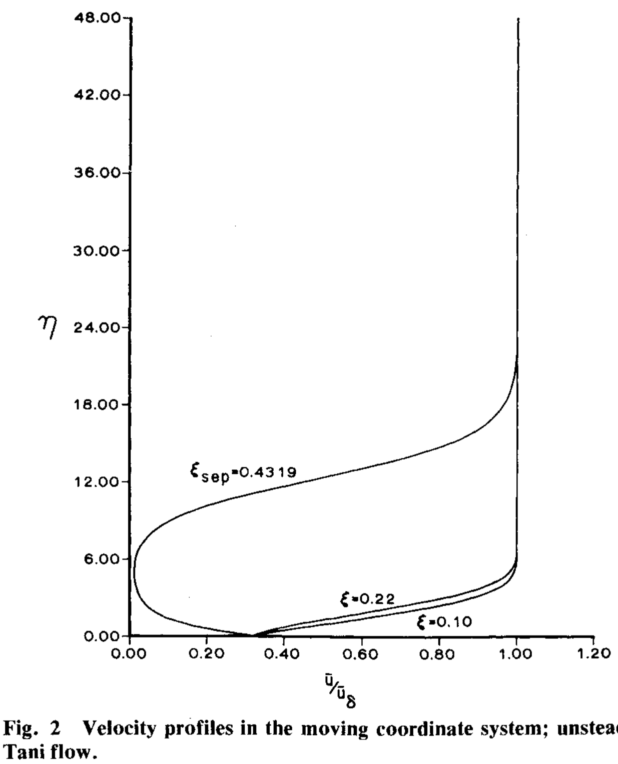

Fig. 2 Velocity profiles in the moving coordinate system; unsteady Tani flow.

$$\eta(\xi)=\frac{(\gamma-1)M_\infty^2}{1+[(\gamma-1)/2]M_\infty^2}\frac{\beta\xi}{H_\delta(\xi)u_\delta^*}(\beta\xi+u_\delta^*)\frac{\mathrm{d}u_\delta^*}{\mathrm{d}\xi} \tag{13j}$$

$$\ell(\xi)=\frac{c(\xi)}{H_\delta^*}\frac{\mathrm{d}H_\delta^*}{\mathrm{d}\xi}=c(\xi)L(\xi) \tag{13k}$$

Here $M_\infty=U_\infty^2/\gamma RT_\infty$, and C is a constant of integration. Thus, all of the coefficients are defined once one chooses a functional form for $u_\delta^*(\xi)$, the external velocity distribution, and a value for the parameter M_∞, which is a measure of the compressibility effects in the flow. Equations (7) and (8) are formulated for numerical solution by converting the third-order equation, Eq. (7), into two equations—one second-order and one first-order. This conversion leads to the new set of equations:

$$W_1=\frac{\partial f}{\partial\eta} \tag{14a}$$

$$\frac{\partial^2 W_1}{\partial\eta^2}+\alpha_{11}\frac{\partial W_1}{\partial\eta}+\alpha_{12}W_1+\alpha_{13}=\alpha_{14}\frac{\partial W_1}{\partial\xi} \tag{14b}$$

$$\frac{\partial^2 W_2}{\partial\eta^2}+\alpha_{12}\frac{\partial W_2}{\partial\eta}+\alpha_{22}W_2+\alpha_{23}=\alpha_{24}\frac{\partial W_2}{\partial\xi} \tag{14c}$$

in which $W_2=\theta$, and

$$\alpha_{11}=\frac{1}{\lambda}\left\{(d+e)f+\frac{b}{2}\eta+h\frac{\partial f}{\partial\xi}+\frac{\partial\lambda}{\partial\eta}\right\}$$

$$\alpha_{12}=-\frac{1}{\lambda}\left\{d\frac{\partial f}{\partial\eta}+a\right\}$$

$$\alpha_{13}=\frac{1}{\lambda}\left[\frac{\theta-m\left(\frac{\partial f}{\partial\eta}\right)^2}{1-m}\right]\{d+a\}$$

$$\alpha_{14}=-\left(c+h\frac{\partial f}{\partial\eta}\right)$$

$$\alpha_{21}=\frac{Pr}{\lambda}\left\{(d+e)f+\frac{b}{2}\eta+h\frac{\partial f}{\partial\xi}+\frac{1}{Pr}\frac{\partial\lambda}{\partial\eta}\right\}$$

$$\alpha_{22}=\frac{Pr}{\lambda}\left\{\frac{n}{1-m}-\left(c+h\frac{\partial f}{\partial\eta}\right)L\right\}$$

$$\alpha_{23}=(Pr-1)m\left\{\left(\frac{\partial^2 f}{\partial\eta^2}\right)^2+\frac{\partial f}{\partial\eta}\frac{\partial^3 f}{\partial\eta^3}+\frac{\partial f}{\partial\eta}\frac{\partial^2 f}{\partial\eta^2}\frac{1}{\lambda}\frac{\partial\lambda}{\partial\eta}\right\}-\frac{Pr}{\lambda}m\frac{n}{1-m}\left(\frac{\partial f}{\partial\eta}\right)^2$$

$$\alpha_{24}=Pr\alpha_{14}$$

Stewartson[10] has pointed out that for equations which have the form of Eqs. (14a) and (14b), one can expect a singularity in the solution of the equations at the point where the coefficients of the derivative with respect to ξ vanishes; i.e., at the point where $\alpha_{14}(\alpha_{24})$ vanishes. Williams[11] has pointed out that for incompressible flow, the vanishing of the coefficient α_{14} does indeed correspond to a singularity that is associated with either unsteady separation if the pressure gradient is adverse or a Stewartson singularity at the upper edge of the boundary layer if the pressure gradient is favorable. In the next section, we will determine if the connection between the vanishing of α_{14} and unsteady separation can be carried over to compressible flow.

Unsteady Retarded Flow

As an example of the application of the above analysis we consider the problem of unsteady retarded flow in which the velocity outside the boundary layer is given by

$$u_\delta^*=\left[1-\left(\frac{x^*}{1-\beta t^*}\right)^2\right]=1-\xi^2 \tag{15}$$

The steady flow equivalent for this flow is the classical retarded flow first studied by Tani.[12] Using this velocity distribution the coefficients in Eqs. (13a-k) are uniquely defined. In this case we evaluate the constant of integration C so that $p^*=T_\delta^*=\rho_\delta^*=1$ at $\xi=0$. In addition, we take the reference state for the linear viscosity law to correspond to conditions as $x=0$.

With all the coefficients of Eqs. (13) now defined, the solutions to Eqs. (7) and (8) subject to the boundary conditions given by Eq. (10) are easily obtained using a straightforward implicit finite difference technique. Equations (7) and (8) have similar solutions at $\xi=0$ so that the solution procedure is easily started. The solution was marched downstream from $\xi=0$. The problem is nonlinear in that coefficients α_{11}, α_{12}, α_{13}, α_{14}, α_{21}, α_{22}, α_{23}, and α_{24} contain the unknown velocity or temperature functions. At each station the solution is obtained iteratively. For the first iteration, the velocity and total enthalpy profiles are assumed to be the same as the converged profiles from the previously computed station. In each subsequent iteration, the coefficients α_{ij} ($i=1,2$; $j=1,2,3,4$) are evaluated using the results of the last iteration. A converged solution is assumed when the difference between the results at the current iteration and those of the last iteration are less than 10^{-6} of the results for the last iteration. This procedure is rather standard.

We consider first the case in which the Prandtl number is unity and the wall is insulated. For this case, results have been obtained for values of the unsteadiness parameter β [see Eq. (15)] of 0, 0.5, and 1.0 and from values of the freestream Mach number of 0, 1, 2, and 4. The $\beta=0$, $M_\infty=0$ case corresponds to the Tani solution, as it should. The general results of all cases are similar, therefore, we will discuss only the results for $M_\infty=2$, $\beta=1$ in detail. As noted above, the solution is marched downstream from $\xi=0$. As ξ approaches a certain value of ξ, which we will denote as ξ_s, the number of iterations required for convergence increases rapidly. This is generally taken as a sign that the solution is approaching a singularity. At the same time, the minimum value of α_{14} begins to decrease sharply and approaches zero as $\xi\to\xi_s$. Figure 1 shows the variation of the minimum value of α_{14} ($=\alpha_{24}$) for $\beta=1$ for each of the Mach numbers considered. For $M_\infty=2$, $\beta=1$, $\xi_s\simeq0.4319$. This behavior is consistent with the approach to a singularity in the incompressible case[11] and with the arguments above relating the singularity to the vanishing of α_{14}. Therefore, we consider, from the behavior of the solution, that the solution is approaching a singularity, where $\alpha_{14}\to0$, as $\xi\to\xi_s$.

The question now arises as to whether or not the flow in the vicinity of the singularity has the properties postulated in the Moore-Rott-Sears model for unsteady separation. According to this model, separation is most easily recognized in the coordinate system moving with separation. For the present problem, if separation occurs at ξ_s, then it moves with a velocity

$$\left.\frac{\mathrm{d}x^*}{\mathrm{d}t^*}\right|_{\xi_s}=-\beta\xi_s$$

The velocities in the fixed coordinate system are related to those in the moving system by

$$\bar{u}^* = \frac{\bar{u}}{\bar{u}_\delta} = \left[\frac{\partial f}{\partial \eta} + \frac{\beta \xi_s}{u_\delta(\xi)}\right] \bigg/ \left[1 + \frac{\beta \xi_s}{u_\delta(\xi)}\right]$$

where $\bar{u}$ is the velocity and $\bar{u}_\delta$ the velocity of the edge of the boundary layer in the moving coordinate sytem. The velocity profiles in the moving coordinate system are shown in Fig. 2 for the case $M_\infty = 2$, $\beta = 1$. Clearly, as the singularity is approached, i.e., $\xi \to \xi_s$, the velocity profile in the moving coordinate system approaches the velocity profile for unsteady separation as postulated in the Moore-Rott-Sears model. Similar results were obtained in all cases studied and, therefore, it is concluded that the Moore-Rott-Sears model for unsteady separation is valid for compressible as well as incompressible flow.

Finally, we consider the effect of Mach number on the separation location. The value of ξ at separation, ξ_s, is shown as a function of Mach number M_∞ for $\beta = 0$, 0.5, and 1.0 in Fig. 3. As in the steady case ($\beta = 0$), the instantaneous separation point is closer to the leading edge ($x = 0$) at higher Mach numbers. It is important to note, however, that in contrast to the steady case ($\beta = 0$), in the unsteady cases $\beta = 0.5$ and $\beta = 1.0$ the separation point is in motion forward along the body. The instantaneous location of separation is given by $x_s(t^*) = \xi_s(1 - \beta t^*)$.

The above results were obtained for the case of an insulated wall. This wall boundary condition was replaced with the condition of a wall at a fixed temperature, and the problem was restudied. Again, a Prandtl number of unity was assumed and the reference state in the viscosity law was assumed to be the state at $x = 0$, denoted by the subscript ∞. Basically, the results were the same as noted above. A singularity occurred in the solution as the coefficient α_{14} ($= \alpha_{24}$) approached zero. The velocity profiles near the singularity as seen in a coordinate system moving with the singularity exhibited just the characteristics postulated in the Moore-Rott-Sears model for separation, verifying the model and the postulation that the singularity in the boundary-layer solution is assoicated with separation. The only difference between the adiabatic wall and fixed wall temperature solutions was the variation of separation location with Mach number. For the fixed wall temperature case, the instantaneous separation location is further downstream at higher Mach numbers, as indicated in Fig. 3. Again, it is important to remember that for unsteady separation ($\beta = 0.5$ and 1.0) the separation point is moving upstream along the wall.

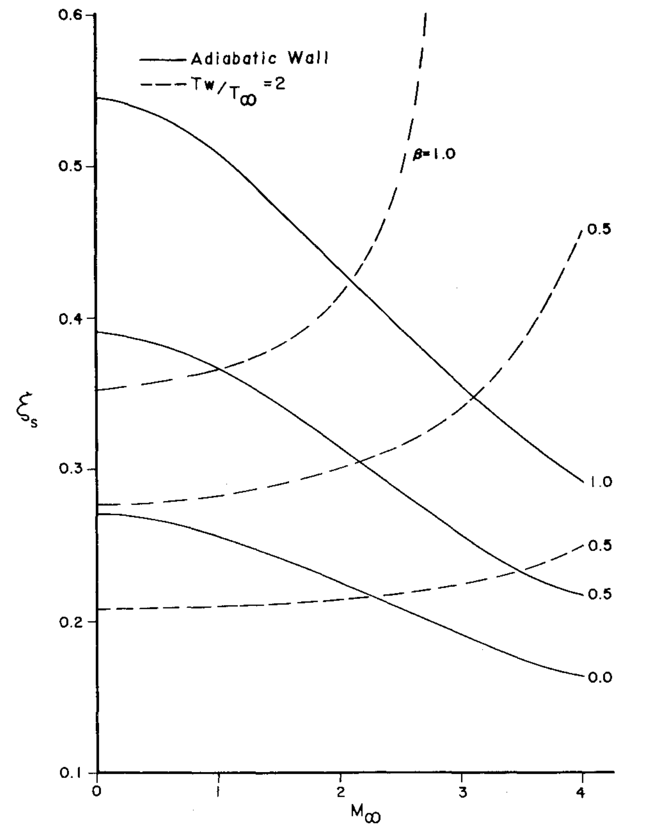

Fig. 3 Location of separation as a function of Mach number; unsteady Tani flow.

Curle's Cubic Velocity

As a further check on the nature of unsteady separation for compressible flow, the theory of semisimilar solutions developed herein was applied to an unsteady variation of Curle's cubic velocity distribution given by

$$u_\delta^* = (3\sqrt{3}/2)(\xi - \xi^3)$$

where again $\xi = x^*/(1 - \beta t^*)$. With $\beta = 0$ this is the steady flow velocity distribution studied by Curle.[13] For $\beta \neq 0$ the external velocity distribution is clearly unsteady. Using this velocity distribution the coefficients in Eqs. (13a-g) are uniquely defined and the solution to Eqs. (7) and (8) may proceed as in the case of Tani's retarded flow. In the present case, the reference state is taken at the point of maximum velocity; i.e., $\xi = 1/\sqrt{3}$ where the reference velocity and temperature are U_∞ and T_∞, respectively.

Results have again been obtained for values of the unsteadiness parameters β of 0, 0.5, and 1.0 for Mach numbers of 0, 1, 2, and 4 for Prandtl numbers of 1.0 and 0.72 and for the adiabatic wall case and the case where $T_w/T_\infty = 2$. In general, the results of all cases studied are similar; therefore, only the results for the adiabatic wall case with $M_\infty = 2$, $\beta = 1$ and $Pr = 0.72$ will be presented. As in the case of Tani's retarded flow, the solution is marched in the direction of increasing ξ. As ξ approaches a certain value of ξ, denoted ξ_s, the number of iterations required for convergence increases rapidly. As before, this is taken as a sign of the approaching

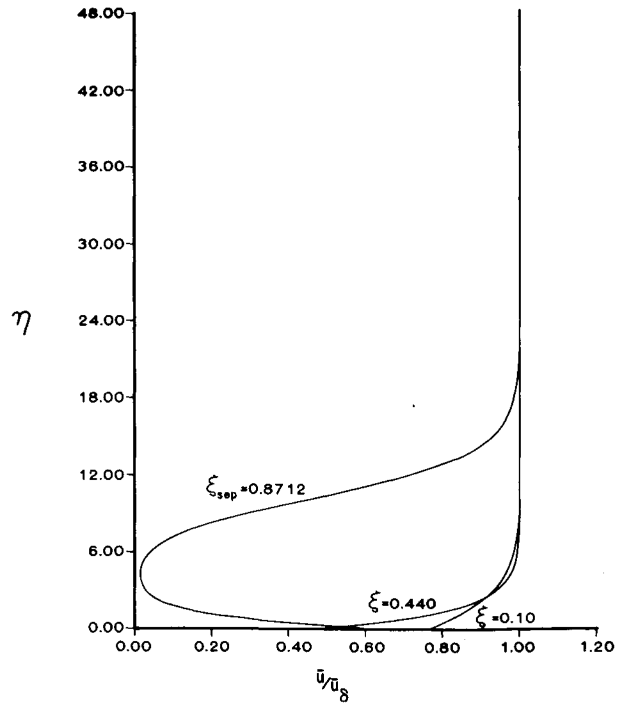

Fig. 4 Velocity profiles in the moving coordinate system; unsteady Curle flow.

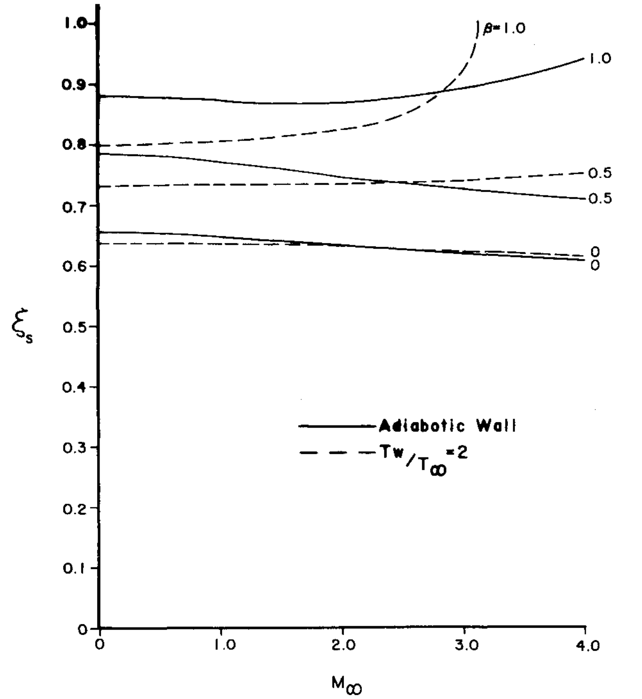

Fig. 5 Location of separation as a function of Mach number; unsteady Curle flow.

of a singularity. The value of α_{14} increases and then drops precipitously as in the earlier case. For $M_\infty = 2.0$, $\beta = 1$, $\xi_s \simeq 0.8712$. Here again the velocity profiles in the coordinate system moving with the singularity approach the separation profile as postulated in the Moore-Rott-Sears model as shown in Fig. 4. These results again verify the Moore-Rott-Sears model as well as the postulation of a singularity at separation in this model. There appears to be no substantial effect of Prandtl number on this result.

The effect of Mach number on the instantaneous location of separation for both the adiabatic wall case and the fixed wall case is shown in Fig. 5. Here the effect of Mach number is not nearly as pronounced as in Tani's case. For the adiabatic wall case there is a slight decrease in ξ_s with Mach number for $\beta = 0$, but a slight increase in ξ_s with Mach number for $\beta = 0.5$ and 1.0. Similar results are obtained in the constant wall temperature case. For $M_\infty > 2$ and $\beta = 1$, the separation point moves toward the rear stagnation point with increasing Mach number. Beyond $M_\infty = 3.1$, the separation does not occur ahead of the rear stagnation point defined by $\xi = 1$.

Linearly Retarded Flow

Calculations were also made for the case of an unsteady counterpart of Howarth's linearly retarded flow, $u_\delta^* = 1 - \xi$, $\xi = x^*/(1 - \beta t^*)$. In the process of carrying out this work, the authors were informed that this particular unsteady flow had been studied earlier by Kim.[14] The results obtained in the present study duplicate, to some extent, the results obtained by Kim and, therefore, are not presented in detail herein. However, Kim did not address the correlation of the vanishing of the coefficient α_{14}. It suffices to note that in the present study the correlation between the vanishing of α_{14} and separation, and the verification of the Moore-Rott-Sears model were obtained for the unsteady variation of Howarth's linearly retarded flow as well as for the unsteady variations of Tani's retarded flow and Curle's flow.

Concluding Remarks

The theory of semisimilar solutions to unsteady boundary-layer equations developed previously for incompressible laminar flows has been extended to the case of compressible laminar flows. This theory has been applied to three separate external velocity distributions which induce separation and for several values of Mach number, Prandtl number, and the unsteadiness parameter, as well as to two wall boundary conditions. The objective of this study has been to investigate the nature of compressible unsteady laminar separation.

All of the results obtained indicate that a singularity occurs in the solution when a certain coefficient, α_{14}, vanishes. It is further shown that the velocity profiles for the flow approach the velocity profiles postulated in the Moore-Rott-Sears model for unsteady laminar separation. Therefore, it is concluded that the Moore-Rott-Sears model is valid for compressible as well as incompressible flow and, further, that there is a singularity in the solution to the boundary-layer equations associated with unsteady separation.

Acknowledgment

This work was supported in part by Grant AEES-82-N1-WMS from Auburn University Engineering Experiment Station.

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